HOMOGENIZATION OF
MAGNETO-ELECTRO-ELASTIC MULTILAMINATED
MATERIALS

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Summary

In this work, based on the periodic unfolding homogenization technique, the limiting equations modelling the behaviour of three-dimensional magneto-electro-elastic periodic structures are rigorously established. The local problems and the corresponding homogenized coefficients of the elastic, dielectric, magnetic permittivity, piezoelectric, piezomagnetic and magneto-electric (ME) tensors are explicitly described. The homogenization model is exemplified for laminated composites and a unified general formula for all effective properties of periodic multilaminated magneto-electro-elastic composites is obtained. This formula is applied to investigate the global behaviour for the important case of transversely isotropic constituents and any finite number of layers in each periodic cell. Examples that provide theoretical evidence of the presence of both a product property and the ME effect are given.

1. Introduction

The magneto-electric (ME) effect is a coupled effect in which application of a magnetic field induces an electric polarization and an electric field induces magnetization. Composites made of piezoelec-

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tric and piezomagnetic components can exhibit such an effect when it is absent from their individual constituents. Avellaneda and Harshé (1) showed that laminated composites can be designed from piezoelectric and magnetostrictive materials, the latter working in the linear range, in order to obtain a larger ME response than that exhibited by other types of composites. In particular, multilaminated magneto-electro-elastic structures have been the subject of interest in recent studies (both theoretical and experimental) of ME interactions and their applications in smart materials/intelligent structures systems (2 to 11).

During the last few years, several homogenization techniques have been applied to investigate the global behaviour of magneto-electro-thermo-elastic composites. For instance, Li and Dunn (12) developed a micromechanical methodology based on the Mori and Tanaka (13) mean field approach combined with the Eshelby tensor, obtained in (14), to derive explicit formulae for the effective properties of two technologically important classes of composites: continuous cylindrical fibres and layers. Also, Huang and Kuo (15) and Wu and Huang (16) used the Mori–Tanaka method, and the magneto-electro-elastic Eshelby tensors, to obtain explicit formulae for the ME in piezomagnetic composites with ellipsoidal inclusions. Aboudi (17) presented a general micromechanical homogenization theory for multiphase magneto-electro-thermo-elastic materials with periodic inclusions of finite dimensions. His results are applied to special cases of fibrous and laminated composites and are compared with those derived from the above-mentioned approach of Li and Dunn (12) and the generalized method of cells (18). Chen et al. (19), using a micromechanical model, gave a procedure to evaluate the effective properties of laminated composites. Lee et al. (20) combined a finite-element analysis with a micromechanical model, based on a representative volume method, in order to study the electromagnetic effective behaviour of two- and three-phase fibrous composites.

Miara et al. (21) combined the multi-scale homogenization method with the finite-element method to illustrate the influence of the shape and orientations of the inclusions on the homogenized characteristics in periodic elastic, piezoelectric and electromagnetic composites. Camacho-Montes et al. (22), using the asymptotic homogenization model and elements of analytic function theory, derived closed-form formulae for magneto-electro-elastic unidirectional fibrous composites with circular cross-section fibres and transversely isotropic magneto-electro-elastic phases. These results represent an application of homogenization theory that is rigorously established in the present contribution for a wider class of such composite materials.

In this paper, the general homogenization theory is applied to the special case of laminated materials and all effective properties of periodic multilaminated magneto-electro-elastic composites are explicitly given. The general formulae obtained are a generalization of those in (23), with the particularity that they are presented here in a unified way which is better for their computational implementation. This general formula is specified for two examples of multilaminated composites with transversely isotropic magneto-electro-elastic constituents to obtain all their effective coefficients. In these examples, matrices that contain the significant magneto-electro-elastic effective coefficients are identified. Such matrices (which are ‘harmonic mean’ type, that is, $\langle M^{-1} \rangle^{-1}$) allow the derivation of new relationships of proportionality between the relevant overall moduli. Numerical calculations and comparisons with other theories are also presented or discussed.

2. Setting of the problem

In this section, we describe a structure composed of a three-dimensional (3D) body made of a magneto-electro-elastic material that is generated as a periodic lattice of the reference (periodic)
cell. The homogenization results are described in the framework of static deformations. However, they are also useful in the long-wavelength regime; see section 7.

2.1 Geometry of the structure

Let $\Omega$ be an open bounded subset of $\mathbb{R}^3$ with a Lipschitz boundary $\Gamma = \partial \Omega$ ($\overline{\Omega}$ is the reference configuration which is assumed to be stress free). We denote by $\varepsilon > 0$, the size of each microstructure which is intended to go to zero. Let $Y = [0, 1]^3$ be a reference (elementary) cell and let $\Gamma_Y = \partial Y$. For any $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, we let $Y^k = k + Y$ and $\Gamma_k = k + \Gamma_Y$. For any fixed $\varepsilon$, let $K_\varepsilon$ be the set of the triplets such that $\varepsilon Y^k$ is included in $\Omega$, namely, $K_\varepsilon := \{k \in \mathbb{Z}^3, \varepsilon Y^k \cap \Omega \neq \emptyset\}$. We suppose that

$$\partial \Omega \bigcap \left( \bigcup_{k \in K_\varepsilon} (\varepsilon \Gamma_k) \right) = \emptyset. \quad (2.1)$$

Clearly, the hypothesis (2.1) implies that near $\partial \Omega$ the material will not behave as an effective material with homogenized coefficients (24); see Fig. 1.

2.2 Constitutive laws and equilibrium equations of magneto-electro-elastic materials

The properties of a 3D body made of a magneto-electro-elastic material are described by six tensors: the elasticity tensor $c_{ijkl}$, the dielectric permittivity tensor $\kappa_{ij}$, the magnetic permittivity tensor $\mu_{ij}$, the ME coupling tensor $\alpha_{ij}$, the piezoelectric coupling tensor $e_{ijk}$ and the piezomagnetic coupling tensor $q_{ijk}$ (Latin indices and exponents take their values in the set $\{1, 2, 3\}$ and Greek ones (except $\varepsilon$) take their values in the set $\{1, 2\}$. Einstein summation convention is used, bold symbols represent vector fields or vector spaces). Thus, for $\varepsilon > 0$, the material functions just introduced are

\[ fig. 1 \] Description of a periodic domain and the unit cell
In a standard way, we obtain the weak (variational) formulation of problem (2.2), (2.3):

\[ c^{ijkl}_e(x) = c^{ijkl}(x/\varepsilon), \quad \varepsilon^{ijkl}_e(x) = \varepsilon^{ijkl}(x/\varepsilon), \quad q^{ijkl}_e(x) = q^{ijkl}(x/\varepsilon), \quad \kappa^{ij}_e(x) = \kappa^{ij}(x/\varepsilon), \quad \alpha^{ij}_e(x) = \alpha^{ij}(x/\varepsilon), \quad \mu^{ij}_e(x) = \mu^{ij}(x/\varepsilon). \]

The tensors of material functions satisfy the usual symmetry conditions

\[ c_{ijkl}^{ijkl} = c_{ijlk}^{ijkl} = c_{klji}^{ijkl}, \quad \varepsilon_{ijkl} = \varepsilon_{ijlk} = \varepsilon_{klji}, \quad q_{ijkl} = q_{ijlk} = q_{klji}, \quad \kappa_{ijkl} = \kappa_{ijlk} = \kappa_{klji}, \quad \alpha_{ijkl} = \alpha_{ijlk} = \alpha_{klji}, \quad \mu_{ijkl} = \mu_{ijlk} = \mu_{klji}. \]

We make the following assumptions:

\[ \exists \eta_1 > 0, \forall \varepsilon > 0, \forall X \in \mathbb{R}^3, \quad c_{ijkl}^{ijkl} X_{ij} X_{kl} \geq \eta_1 X_{ij} X_{ij}, \]
\[ \exists \eta_2 > 0, \forall \varepsilon > 0, \forall a \in \mathbb{R}^3, \quad \kappa_{ijkl} a_i a_j \geq \eta_2 a_i a_i, \]
\[ \exists \eta_3 > 0, \forall \varepsilon > 0, \forall b \in \mathbb{R}^3, \quad \mu_{ijkl} b_i b_j \geq \eta_3 b_i b_i, \]

where \( \mathbb{R}^3 \) is the space of symmetric 3 \( \times \) 3 matrices.

For a fixed \( \varepsilon > 0 \), the magneto-electro-mechanical behaviour of the body is given by the elastic displacement field \( u_e = (u^e_i): \Omega_e \rightarrow \mathbb{R}^3 \), the electric potential \( \varphi_e: \Omega_e \rightarrow \mathbb{R} \) and the magnetic potential \( \psi_e: \Omega_e \rightarrow \mathbb{R} \), which solve the following equilibrium equations:

\[ \text{div} \sigma_e(u_e, \varphi_e, \psi_e) = -f, \quad \text{div} D_e(u_e, \varphi_e, \psi_e) = 0, \quad \text{div} B_e(u_e, \varphi_e, \psi_e) = 0 \quad \text{in} \ \Omega, \quad (2.2) \]

with \( f \in L^2(\Omega), \sigma_e = (\sigma_{ij}^e), D_e = (D^i_e), B_e = (B^i_e), (\text{div} \sigma_e)^i = \partial_j \sigma_{ij}^e, \text{div} D_e = \partial_i D^i_e, \text{div} B_e = \partial_i B^i_e, \partial_i = \partial / \partial x_i, x = (x_i) \in \Omega \) and \( s_{kl}(u_e) = \frac{1}{2} (\partial_k u^e_l + \partial_l u^e_k) \).

The stress tensor \( \sigma_e \), the electric displacement \( D_e \) and the magnetic displacement \( B_e \) are related to the linearized strain \( s_{kl}(u_e) \) and to the gradients of the electric \( (\partial_k \varphi_e) \) and magnetic \( (\partial_k \psi_e) \) potentials through the constitutive laws:

\[ \sigma_{ij}^e(u_e, \varphi_e, \psi_e) = c_{ijkl}^e s_{kl}(u_e) + \varepsilon_{ijkl}^e \partial_m \varphi_e + q_{ijkl}^e \partial_n \psi_e, \]
\[ D^i_e(u_e, \varphi_e, \psi_e) = c_{ijkl}^e s_{kl}(u_e) - \kappa_{ijkl}^e \partial_m \varphi_e - \alpha_{ijmn}^e \partial_n \psi_e, \]
\[ B^i_e(u_e, \varphi_e, \psi_e) = q_{ijkl}^e s_{kl}(u_e) - \alpha_{ijmn}^e \partial_n \psi_e - \mu_{ijkl}^e \partial_n \psi_e. \quad (2.3) \]

To complete the formulation, boundary conditions have to be supplied. To ease the explanation, we consider the homogeneous Dirichlet boundary conditions on the external boundary \( \partial \Omega \), that is, \( u_e = 0, \varphi_e = 0 \) and \( \psi_e = 0 \) on \( \partial \Omega \); however, general mixed-type conditions could be assumed.

### 2.3 Properties of a 3D body made of magneto-electro-elastic material

In a standard way, we obtain the weak (variational) formulation of problem (2.2), (2.3):

Find \( (u_e, \varphi_e, \psi_e) \in H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega) \) such that

\[ a_e((u_e, \varphi_e, \psi_e); (v_e, \varphi_e, \theta_e)) = L_e(v_e, \varphi_e, \theta_e). \quad (2.4) \]
3. Preliminaries for the periodic unfolding method

In order to find the effective (macroscopic) behaviour of a magneto-electro-elastic solid with microperiodic structure, we employ the periodic unfolding method (25). This method sheds new light on the so-called two-scale convergence but is much more elementary and straightforward. It renders the proof of homogenization results quite elementary (making use of simple weak convergence problems in a Lebesgue space). It also provides error estimates and corrector results (for details, see (25)).

We denote by \( x \in \Omega \) the slow variable (outer or macro-variable) and by \( y = x/\varepsilon \in Y \) the rapid variable (inner or micro-variable). For each \( x \in \mathbb{R}^3 \) and \( \varepsilon > 0 \), we have the unique decomposition \( x = \varepsilon ([x/\varepsilon]_Y + [x/\varepsilon]_Y) \), where for \( z \in \mathbb{R}^3 \), \([z]_Y\) is the unique integer combination such that \([z]_Y = z - [z]_Y\) belongs to the cell \( Y \). We now introduce the unfolding operator \( T_\varepsilon \) and the averaging operator \( U_\varepsilon \) related to the study of periodic structures.

**Definition 3.1.** The periodic unfolding operator \( T_\varepsilon : L^2(\Omega) \to L^2(\Omega \times Y) \) is defined by

\[
\forall w \in L^2(\Omega), \quad T_\varepsilon(w)(x, y) = w(\varepsilon [x/\varepsilon]_Y + \varepsilon y) \quad \text{for each} \quad (x, y) \in \Omega \times Y.
\]

**Definition 3.2.** The averaging operator \( U_\varepsilon : L^2(\Omega \times Y) \to L^2(\Omega) \) is defined by

\[
U_\varepsilon(\Phi) = \frac{1}{|Y|} \int_Y \Phi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\}_Y \right) \, dz \quad \text{for} \quad \Phi \in L^2(\Omega \times Y).
\]

The main properties of these operators are given below (see (25) for complete proofs).

**Lemma 3.3.** (Cioranescu et al. (25))

(i) For all \( v, w \in L^2(\Omega) \), we have the properties

\[
T_\varepsilon(w)(x, [x/\varepsilon]_Y) = w(x), \quad T_\varepsilon(vw) = T_\varepsilon(v)T_\varepsilon(w), \quad T_\varepsilon(v + w) = T_\varepsilon(v) + T_\varepsilon(w).
\]

(ii) Let \( v \in L^2(\Omega) \). Then, \( T_\varepsilon(v) \to v \) strongly in \( L^2(\Omega \times Y) \).

(iii) Let \( \{v_\varepsilon\}_\varepsilon \) be a uniformly bounded sequence in \( L^2(\Omega) \). Then, there exists \( v \in L^2(\Omega \times Y) \) such that \( T_\varepsilon(v_\varepsilon) \to v \) weakly in \( L^2(\Omega \times Y) \).

(iv) Let \( \{v_\varepsilon\}_\varepsilon \) be a uniformly bounded sequence in \( H^1(\Omega) \). Then, there exist two fields, \( v \in H^1(\Omega) \) and \( \bar{v} \in L^2(\Omega, H^1_\varepsilon(Y)) \), such that

\[
T_\varepsilon(v_\varepsilon) \rightharpoonup v \quad \text{weakly in} \quad L^2(\Omega \times Y) \quad \text{and} \quad T_\varepsilon(\nabla_x v_\varepsilon) \to \nabla_x v + \nabla_y \bar{v} \quad \text{weakly in} \quad L^2(\Omega \times Y, \mathbb{R}^3),
\]

Under classical regularity assumptions, \( f \in L^2(\Omega) \) and material functions in \( L^\infty(\Omega) \), the variational problem (2.4), (2.5) has a unique solution \( (u_\varepsilon, \phi_\varepsilon, \psi_\varepsilon) \).
where the subscripts $x$ and $y$ represent derivation with respect to the first variable $x$ and the second variable $y$, respectively. Also, $H^1_y(Y)$ denotes the Sobolev space of $Y$-periodic square-integrable functions up to their first generalized derivative defined in $Y$ ($Y$-periodic functions attain the same function values at corresponding points on opposite sides of $\partial Y$).

(v) If $\{T_e(v_e)\}_{e}$ strongly converges to $v$ in $L^2(\Omega \times Y)$, then $v_e - U_e(v) \to 0$ in $L^2(\Omega)$.

4. Homogenization of the problem

Now, we apply the periodic unfolding technique to the magneto-electro-elastic problem. We assume that the periodic material functions $e_{ijkl}(y)$, $e^k(y)$, etc. are of class $C^\infty(Y)$. The solution of the variational problem (2.4), (2.5) satisfies the a priori uniform estimate

$$\|u_e\|_{H^1(y)}^2 + \|\varphi_e\|_{H^1(\Omega)}^2 + \|\psi_e\|_{H^1(\Omega)}^2 \leq C,$$

(4.1)

with a constant $C$ which depends upon $\Omega$ and $Y$ but which is independent of $e$. This is a consequence of the Korn and Poincaré inequalities for periodic domains.

In Lemma 4.1, we present the main asymptotic properties of the periodic unfolding method applied to magneto-electro-elastic problems.

**Lemma 4.1.** Let $u_e$, $\varphi_e$ and $\psi_e$ solve (2.2), (2.3). Then, there exists a unique limit magneto-electro-elastic field $(u, \varphi, \psi) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ and three corrector fields $(\tilde{u}, \tilde{\varphi}, \tilde{\psi}) \in L^2(\Omega, H^1_y(Y)/\mathbb{R}) \times L^2(\Omega, H^1_y(Y)/\mathbb{R}) \times L^2(\Omega, H^1_y(Y)/\mathbb{R})$ such that, up to subsequences,

$$T_e(u_e) \rightharpoonup u \quad \text{weakly in } L^2(\Omega \times Y),$$
$$T_e(\varphi_e) \rightharpoonup \varphi, \quad T_e(\psi_e) \rightharpoonup \psi \quad \text{weakly in } L^2(\Omega \times Y),$$
$$T_e(s_{kl,x}(u_e)) \rightharpoonup s_{kl,x}(\tilde{u}) \quad \text{weakly in } L^2(\Omega \times Y),$$
$$T_e(\nabla_x \varphi_e) \rightharpoonup \nabla_x \varphi + \nabla_y \tilde{\varphi} \quad \text{weakly in } L^2(\Omega \times Y),$$
$$T_e(\nabla_x \psi_e) \rightharpoonup \nabla_x \varphi + \nabla_y \tilde{\psi} \quad \text{weakly in } L^2(\Omega \times Y).$$

**Proof.** The convergence result is a consequence of the uniform bound (4.1) and the properties of the periodic unfolding method (Lemma 3.3).

We remark that $(u, \varphi, \psi)$ and $(\tilde{u}, \tilde{\varphi}, \tilde{\psi})$ can be interpreted with the following asymptotic expansions: $u_e(x) = u(x) + e\tilde{u}(x, y) + \cdots$, $\varphi_e(x) = \varphi(x) + e\tilde{\varphi}(x, y) + \cdots$ and $\psi_e(x) = \psi(x) + e\tilde{\psi}(x, y) + \cdots$.

In the sequel, we assume that the constitutive parameters are periodic. We are now in a position to exhibit the behaviour of the limit fields.

**Lemma 4.2.** The fields $(u, \varphi, \psi)$ and $(\tilde{u}, \tilde{\varphi}, \tilde{\psi})$ solve the following variational problem: for all $v \in H^1_0(\Omega)$, $v^1 \in L^2(\Omega; H^1_y(Y)/\mathbb{R})$, $\phi \in H^1_0(\Omega)$, $\phi^1 \in L^2(\Omega; H^1_y(Y)/\mathbb{R})$, $\theta \in H^1_0(\Omega)$ and $\theta^1 \in L^2(\Omega; H^1_y(Y)/\mathbb{R})$,

$$\int_{\Omega \times Y} \left\{ e^{ijkl}(s_{ijkl}(u) + s_{ijkl}(\tilde{u})) + e^{mkl}(\tilde{\epsilon}_{m,x} \varphi + \tilde{\epsilon}_{m,y} \tilde{\varphi}) + q^{mkl}(\tilde{\epsilon}_{m,x} \psi + \tilde{\epsilon}_{m,y} \tilde{\psi}) \right\}$$
$$\times (s_{kl,x}(v) + s_{kl,y}(v^1))dxdy = \int_{\Omega} f \cdot vdx,$$
\[
\begin{align*}
\int_{\Omega \times Y} \{ & e^{mkl}(s_{kl,x}(\mathbf{u}) + s_{kl,y}(\mathbf{u})) - \kappa^{mi}(\hat{c}_{i,x}\varphi + \hat{c}_{i,y}\bar{\varphi}) - \alpha^{mi}(\hat{c}_{i,x}\psi + \hat{c}_{i,y}\bar{\psi}) \\
& \times (\hat{c}_{m,x}\varphi + \hat{c}_{m,y}\bar{\varphi})\} \, dx \, dy = 0, \\
\int_{\Omega \times Y} \{ & q^{mkl}(s_{ij,x}(\mathbf{u}) + s_{ij,y}(\mathbf{u})) - \alpha^{im}(\hat{c}_{i,x}\varphi + \hat{c}_{i,y}\bar{\varphi}) - \mu^{mi}(\hat{c}_{i,x}\psi + \hat{c}_{i,y}\bar{\psi}) \\
& \times (\hat{c}_{m,x}\theta + \hat{c}_{m,y}\bar{\theta})\} \, dx \, dy = 0,
\end{align*}
\]

where \( \hat{c}_{m,x}\varphi = \hat{\varphi}/\hat{\varphi} x\) and \( s_{rt,x}(\mathbf{u}) = \frac{1}{2}(\hat{c}_{r,x}u_r + \hat{c}_{r,x}u_t) \).

**Proof.** In conjunction with the following properties,

\[ \int_{\Omega} \nu \, d\mathbf{x} = \int_{\Omega \times Y} \mathcal{T}_\varepsilon(v) \, dx \, dy \quad \text{for all } v \in L^1(\Omega), \]

and \( \mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w) \) for all \( v, w \in L^1(\Omega) \), we use in (2.4), (2.5) test functions from distribution spaces, \( v \in \mathcal{D}(\Omega; \mathbb{R}^3) \), \( \phi \in \mathcal{D}(\Omega) \) and \( \theta \in \mathcal{D}(\Omega) \), and we let \( \varepsilon \to 0 \). Next, we take test functions \( v_\varepsilon(x) = \varepsilon v(x, x/\varepsilon) \), \( v \in \mathcal{D}(\Omega, \mathcal{D}'(\varepsilon)(Y; \mathbb{R}^3)) \), \( \phi_\varepsilon(x) = \varepsilon \phi(x, x/\varepsilon) \), \( \phi \in \mathcal{D}(\Omega, \mathcal{D}'(\varepsilon)(Y)) \), and \( \theta_\varepsilon(x) = \varepsilon \theta(x, x/\varepsilon) \), \( \theta \in \mathcal{D}(\Omega, \mathcal{D}'(\varepsilon)(Y)) \), and pass to the limit.

The problem model suggests the expressions of the periodically oscillating functions (the correctors) as

\[
\begin{align*}
\hat{u}(x, y) &= s_{rt,x}(u(x))w^{rt}(y) + \hat{c}_{m,x}\varphi(x)g^m(y) + \hat{c}_{n,x}\psi(x)f^r(y), \\
\hat{\varphi}(x, y) &= s_{rt,x}(u(x))\zeta^{rt}(y) + \hat{c}_{m,x}\varphi(x)\pi^m(y) + \hat{c}_{n,x}\psi(x)\chi^n(y), \\
\hat{\psi}(x, y) &= s_{rt,x}(u(x))\eta^{rt}(y) + \hat{c}_{m,x}\varphi(x)\xi^m(y) + \hat{c}_{n,x}\psi(x)\gamma^n(y),
\end{align*}
\]

where the basis (functions) \( (w^{rt}, \zeta^{rt}, \eta^{rt}) \), \( (g^m, \pi^m, \xi^m) \) and \( (f^r, \chi^n, \gamma^n) \) solve local microscopic problems posed in the elementary cell \( Y \). We note that the superscripts on the local solutions relate to the forcing in the local problems (see Lemma 4.3). Now, both local and homogenized problems can be formulated by means of the following lemma.

**Lemma 4.3.** (Local functions in \( Y \)) There exist unique \( Y \)-periodic corrector basis functions (up to additive constants) with

\[
\begin{align*}
(w^{rt}, \zeta^{rt}, \eta^{rt}) &\in H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}), \\
(g^m, \pi^m, \xi^m) &\in H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}), \\
(f^r, \chi^n, \gamma^n) &\in H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}) \times H^1_2(Y/\mathbb{R}),
\end{align*}
\]

which are weak solutions of the local (microscopic) problems posed in the elementary cell \( Y \).

The six triplets \((\check{w}^{rt}, \check{\zeta}^{rt}, \check{\eta}^{rt})\) are solutions to the following problems:

\[
\begin{align*}
-\tilde{c}_i\sigma^{ij}(\hat{w}^{rt}, \hat{\zeta}^{rt}, \hat{\eta}^{rt}) &= \hat{c}_i\hat{\varepsilon}^{jrt} & \text{on } Y, \\
-\tilde{c}_i\check{D}^i(\hat{w}^{rt}, \hat{\zeta}^{rt}, \hat{\eta}^{rt}) &= \hat{c}_i\hat{\varepsilon}^{irt} & \text{on } Y, \\
-\tilde{c}_i\tilde{B}^i(\hat{w}^{rt}, \hat{\zeta}^{rt}, \hat{\eta}^{rt}) &= \hat{c}_i\hat{q}^{irt} & \text{on } Y.
\end{align*}
\]

(4.4)
The three triplets \((\tilde{g}^m, \pi^m, \zeta^m)\) are solutions to the following problems:

\[
\begin{align*}
-\partial_i \sigma^{ij}(g^m, \pi^m, \zeta^m) &= \tilde{\sigma}^{ij} \epsilon^{mij} & \text{on } Y, \\
-\partial_i D^j(g^m, \pi^m, \zeta^m) &= -\partial_i \kappa^{im} & \text{on } Y, \\
-\partial_i B^i(g^m, \pi^m, \zeta^m) &= -\partial_i \alpha^{im} & \text{on } Y. 
\end{align*}
\]  

(4.5)

The three triplets \((\tilde{f}^m, \chi^m, \gamma^m)\) are solutions to the following problems:

\[
\begin{align*}
-\partial_i \sigma^{ij}(f^m, \chi^m, \gamma^m) &= \tilde{\sigma}^{ij} q^{ijm} & \text{on } Y, \\
-\partial_i D^j(f^m, \chi^m, \gamma^m) &= -\partial_i \alpha^{im} & \text{on } Y, \\
-\partial_i B^i(f^m, \chi^m, \gamma^m) &= -\partial_i \mu^{im} & \text{on } Y. 
\end{align*}
\]  

(4.6)

**Proof.** In (4.2), we take test functions \(v = 0, v^1 = v^1(y), \phi = 0, \phi^1 = \phi^1(y), \theta = 0\) and \(\theta^1 = \theta^1(y)\). Thus, we obtain the main result.

We note that since the data such as \(c_{ijkl}\) are discontinuous functions, the derivatives on the right-hand sides of the above problems must be understood in the distributions sense. We note also that, although the initial problem models an insulator (no free electric or magnetic charges), the local problems contain volume electric and magnetic charges.

The limit displacement, electric and magnetic potentials \((u, \varphi, \psi)\) solve a problem of the same type as (2.2) but with another definition of the homogenized stress, electric and magnetic displacement tensors \(\tilde{\sigma}, \tilde{D}\) and \(\tilde{B}\):

\[
\begin{align*}
\text{div } \tilde{\sigma}(u, \varphi, \psi) &= -f, & \text{div } \tilde{D}(u, \varphi, \psi) &= 0, & \text{div } \tilde{B}(u, \varphi, \psi) &= 0 \quad \text{on } \Omega, 
\end{align*}
\]  

(4.7)

with homogeneous boundary conditions \(u = 0, \varphi = 0\) and \(\psi = 0\) on \(\partial \Omega\). The new constitutive law is given by

\[
\begin{align*}
\tilde{\sigma}^{ij}(u, \varphi, \psi) &= \tilde{\sigma}^{ijkl} s_{kj}(u) + \tilde{\epsilon}^{mij} \partial_m \varphi + \tilde{\eta}^{nij} \partial_n \psi, \\
\tilde{D}^j(u, \varphi, \psi) &= \tilde{f}^{ijkl} s_{kl}(u) - \tilde{\kappa}^{im} \partial_m \varphi - \tilde{\alpha}^{im} \partial_n \psi, \\
\tilde{B}^i(u, \varphi, \psi) &= \tilde{g}^{ijkl} s_{kl}(u) - \tilde{\beta}^{im} \partial_m \varphi - \tilde{\mu}^{im} \partial_n \psi.
\end{align*}
\]  

(4.8)

(4.9)

The homogenized elasticity, dielectric, magnetic and coupling tensors have the following definitions:

\[
\begin{align*}
\tilde{\sigma}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{D}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{B}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{\epsilon}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{\eta}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{\kappa}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\tilde{\alpha}^{ijkl} &= \langle \epsilon^{ijkl} [s_{kl,y}(w^t)] + \delta^{ijkl} \rangle \\
\end{align*}
\]  

(4.10)
The homogenized constitutive law \((4.7), (4.8)\) is of the same kind as the original problem (2.2).

**Lemma 4.4.** The homogenized constitutive law \((4.7), (4.8)\) is of the same kind as the original problem (2.2).

**Proof.** First, let us prove that the homogenized elasticity tensor is positive, that is, \(\tilde{c}^{ijkl} g_{ijkl} g_{kl} > 0\) for all symmetric second-order tensors \(g_{ij} = g_{ji}\). By definition, we get

\[
\tilde{c}^{ijkl} g_{ijkl} g_{kl} = \frac{1}{|Y|} \int_Y (c^{ijkl} g_{ijkl}(s_{kl}(w^t g_{rt}) + \delta_{kl}^r g_{rt}) + e^{ijkl} g_{ijkl}(\zeta^t g_{rt}) + q^{ijkl} g_{ijkl}(\eta^t g_{rt}))dy.
\]

By letting \(w = w^t g_{rt}, \zeta = \zeta^t g_{rt}, \eta = \eta^t g_{rt}\) and \(p_{kl} = \delta_{kl}^r g_{rt} = g_{kl}\), we obtain

\[
\tilde{c}^{ijkl} g_{ijkl} g_{kl} = \frac{1}{|Y|} \int_Y (c^{ijkl} p_{ij}(s_{kl}(w) + p_{kl}) + e^{ijkl} p_{ijkl}(\zeta) + q^{ijkl} p_{ijkl}(\eta))dy.
\]

By linearity of local problem (4.4), we can write, for all \((v, \phi, \theta) \in H^1_Y(Y) \times H^1_Y(Y) \times H^1_Y(Y),

\[
\int_Y (c^{ijkl}(s_{kl}(w) + p_{kl})s_{ij}(v) + e^{ijkl}s_{ij}(v)\partial_k\zeta + q^{ijkl}s_{ij}(v)\partial_k\eta)d\gamma = 0,
\]

\[
\int_Y (\partial_i \phi - \alpha^m \partial_i \zeta + \partial_i \eta\theta)d\gamma = 0,
\]

\[
\int_Y (\partial_i \zeta + \beta^m \partial_i \zeta + \mu^m \partial_i \eta\phi)d\gamma = 0.
\]

(4.11)

By using the variational formulation (4.11), we obtain

\[
\tilde{c}^{ijkl} g_{ijkl} g_{kl} = \frac{1}{|Y|} \int_Y (c^{ijkl} p_{ij} p_{kl} + c^{ijkl} s_{ij}(w) s_{kl}(w) + e^{ijkl} s_{ij}(w) \partial_k\zeta + q^{ijkl} s_{ij}(w) \partial_k\eta)dy
\]

\[
\geq \frac{1}{|Y|} \int_Y (c^{ijkl} p_{ij} p_{kl} + c^{ijkl} s_{ij}(w) s_{kl}(w))dy
\]

\[
= \frac{1}{|Y|} \int_Y c^{ijkl} (s_{ij}(w + p_{ij}) s_{kl}(w + p_{kl}))dy.
\]

We observe that the right-hand side of the inequality is strict (it is easy to show that the linearized deformation tensor cannot be constant for a \(Y\)-periodic displacement field: \(s_{ij}(w) + p_{ij} \neq 0\)).

Next, we prove that \(\tilde{c}^{ijkl} = \tilde{f}^{ijkl}\). Multiplying (4.4) by \((\tilde{g}^m, \pi^m, \tilde{\zeta}^m)\) and integrating by parts, we get a new expression for \(\tilde{c}^{ijkl}\). The same expression is obtained for \(\tilde{f}^{ijkl}\) by multiplying (4.5) by \((\tilde{w}^t, \zeta^t, \eta^t)\).
Finally, using the last four lemmas, we can propose an alternative form of the main convergence theorem.

**Theorem 4.5.** (The existence and uniqueness result and homogenized model) The field \((\mathbf{u}, \varphi, \psi)\) is a unique solution of the homogenized problem, with the new constitutive law which is of the same kind as the initial law,

\[
\begin{align*}
\sigma_{ij}(\mathbf{u}, \varphi, \psi) &= \tilde{c}^{ijkl} s_{kl}(\mathbf{u}) + \tilde{c}^{mij} \tilde{c}_m \varphi + \tilde{q}^{nilj} \tilde{c}_n \psi, \\
\overline{D}_i(\mathbf{u}, \varphi, \psi) &= \tilde{e}^{ijkl} s_{kl}(\mathbf{u}) - \tilde{\kappa}^{im} \tilde{c}_m \varphi - \tilde{\alpha}^{in} \tilde{c}_n \psi, \\
\overline{B}_i(\mathbf{u}, \varphi, \psi) &= \tilde{q}^{ijkl} s_{kl}(\mathbf{u}) - \tilde{\alpha}^{im} \tilde{c}_m \varphi - \tilde{\mu}^{in} \tilde{c}_n \psi.
\end{align*}
\]

The main properties of the averaging operator \(\mathcal{U}_e\) are given in Lemma 4.6.

**Lemma 4.6.** (Cioranescu et al. (25)) Set \(\{v_\varepsilon\} \subset L^2(\Omega)\). The following weak convergences are equivalent:

\[
\begin{align*}
T_e(v_\varepsilon) &\rightharpoonup v \text{ in } L^2(\Omega \times Y) \quad \text{and} \quad v_\varepsilon - \mathcal{U}_e(v) \rightharpoonup v \text{ in } L^2(\Omega).
\end{align*}
\]

The following corollary complements the convergence result by providing strong convergence (corrector result) which is very useful from both theoretical and numerical points of view.

**Corollary 4.7** (Strong convergence)

\[
\begin{align*}
\sigma_{ij}(u_\varepsilon) - \sigma_{ij}(\mathbf{u}) - \mathcal{U}_e(s_{ij}(\tilde{u})) &\to 0 \quad \text{strongly in } L^2(\Omega), \\
\nabla \varphi_\varepsilon - \nabla \varphi - \mathcal{U}_e(\nabla \tilde{\varphi}) &\to 0 \quad \text{strongly in } L^2(\Omega), \\
\nabla \psi_\varepsilon - \nabla \psi - \mathcal{U}_e(\nabla \tilde{\psi}) &\to 0 \quad \text{strongly in } L^2(\Omega).
\end{align*}
\]

The following sections are devoted to the application of the general homogenized model presented above to obtain analytical formulae for the effective coefficients of laminated magneto-electro-elastic composites.

5. **Closed-form expressions for effective coefficients of magneto-electro-elastic periodically multilaminated composites**

The local problems (4.4)–(4.6) can be written in a unified form as follows:

Find \(W^{c't}_b\) being \(Y\)-periodic such that

\[
\begin{align*}
\tilde{c}_m \left( C^{a'mb'n} \tilde{c}_n W^{c't}_b \right) &= -\tilde{e}_m \left( C^{a'mc't} \right),
\end{align*}
\]

where the primed Latin indices run from 1 to 5,

\[
\begin{align*}
C^{imkn} &\equiv \tilde{c}^{imkn}, & C^{4mkn} &\equiv \tilde{e}^{mkn}, & C^{4m4n} &\equiv -k^{mn}, & C^{5mkn} &\equiv \tilde{q}^{mkn}, & C^{5m4n} &\equiv -\alpha^{mn}, \\
C^{5m5n} &\equiv -\mu^{mn}, & W^{rt}_k &\equiv w^{rt}_k, & W^{rt}_4 &\equiv \zeta^{rt}, & W^{rt}_5 &\equiv \eta^{rt}, & W^{4m}_k &\equiv \delta^m_k, & W^{4m}_4 &\equiv \pi^m, \\
W^{4m}_5 &\equiv \xi^m, & W^{5m}_k &\equiv f^m_k, & W^{5m}_4 &\equiv \zeta^m \quad \text{and} \quad W^{5m}_5 &\equiv \gamma^m.
\end{align*}
\]

The local problem must be completed with additional contact conditions on the interfaces between the constituents of the composite of interest.
The effective coefficients can be derived from the following formula:

$$\tilde{C}^{a'mb'i} = \left\langle C^{a'mb'i} \right\rangle + \left\langle C^{a'mc'n} \tilde{c}_n W^{b'i} \right\rangle.$$  \hspace{1cm} (5.2)

The above unified formulation is very convenient for some specific problems. For instance, let us consider the particular case of a laminated magneto-electro-elastic composite, made of cells which are periodically along the \(y_2\)-axis. Each cell may be made of any finite number of homogeneous magneto-electro-elastic layers. The axes of symmetry of each layer are parallel to each other and the \(y_2\)-axis is perpendicular to the layering. In this case, the material functions \(C^{a'mb'i}\) and the local functions \(W^{b'i}\) depend only on the fast variable \(y_2\). Consequently, expressions (5.1) and (5.2) take the form

$$D_2 \left( C^{a'2b'i} D_2 W^{e'i} \right) = -D_2 \left( C^{a'2c'i} \right),$$  \hspace{1cm} (5.3)

$$\tilde{C}^{a'mb'i} = \left\langle C^{a'mb'i} \right\rangle + \left\langle C^{a'mc'i} D_2 W^{b'i} \right\rangle,$$  \hspace{1cm} (5.4)

where \(D_2\) denotes the ordinary derivative in the generalized sense with respect to the \(y_2\)-coordinate. The angular brackets define the average per unit length of the relevant quantity over the periodic cell, that is, \(\langle F \rangle = |Y|^{-1} \int_Y F(y_2) dy_2\), where \(|Y|\) denotes the length of \(Y\). For simplicity, a periodic unit cell \(Y\) has been considered. This is a 1D homogenization problem that consists in finding the one-periodic solution of (5.3), of zero average on \(Y\), and satisfying the usual contact interface conditions (see, for instance, (26, Chapter 5)).

Solving the system of ordinary differential equations defined by (5.3), taking into account perfect bounding conditions at the interfaces, and using (5.4), it is possible to obtain the following general closed-form formulae for all the magneto-electro-elastic effective coefficients:

$$\tilde{C}^{a'mb'i} = \left\langle C^{b'mb'i} \right\rangle + \left\langle C^{b'mc'i} \left( C^{c'2d'2} \right)^{-1} C^{d'2b'i} \right\rangle$$

$$+ \left\langle C^{a'mc'i} \left( C^{c'2d'2} \right)^{-1} \right\rangle \left\langle \left( C^{d'2e'2} \right)^{-1} \right\rangle^{-1} \left\langle \left( C^{e'2f'2} \right)^{-1} C^{f'2b'i} \right\rangle. \hspace{1cm} (5.5)$$

Here, \(\left( C^{a'2b'2} \right)^{-1}\) denotes the components of the inverse matrix of \(C^{a'2b'2}\).

Formula (5.5) is a generalization of formula (1.11) in (26, p. 145), where the purely elastic case was investigated.

After some algebraic manipulations, from (5.5), the following formula for the particular case of a two-laminated magneto-electro-elastic composite was derived:

$$\tilde{C}^{a'mb'i} = \lambda C^{a'mb'i} + (1 - \lambda) C^{a'mb'i} - \lambda (1 - \lambda) [C^{a'mc'i}] B^{-1}_{c'd'} [C^{d'2b'i}], \hspace{1cm} (5.6)$$

where \(\lambda\) is the volume fraction of phase 1; the material coefficients of such a composite are piecewise constants, defined by

$$C^{a'mb'i}(y_2) = \begin{cases} C^{a'mb'i}_1, & y_2 \in (0, \lambda), \\ C^{a'mb'i}_2, & y_2 \in (\lambda, 1), \end{cases}$$

\([C^{a'mc'i}] = C^{a'mc'i}_1 - C^{a'mc'i}_2\), a row vector for \(a'm\) fixed, \([C^{d'2b'i}] = C^{d'2b'i}_1 - C^{d'2b'i}_2\), a column vector for \(b'i\) fixed, \(B^{-1}_{c'd'} = \lambda C^{c'2d'2}_2 - (1 - \lambda) C^{c'2d'2}_1\) and \(B^{-1}_{c'd'}\) is the inverse matrix of \(B_{c'd'}\).
Formula (5.6) is similar to formula (17) of (27, p. 138) for laminated thermo-piezoelectric composites.

In the engineering literature, this type of layer distribution is identified as ‘connectivity in parallel’, and the case connecting lamina oriented in the \(O_{y_1,y_2}\) plane (that is, laminated in the \(y_3\)-direction) is known as ‘connectivity in series’ (28). In the next subsections, formula (5.5) will be specialized in order to obtain explicit formulae for two examples involving both parallel and series connectivity. In both examples, the periodic unit cell can possess any finite number of homogeneous magneto-electro-elastic materials with transversely isotropic properties. Each phase is characterized by the following independent constants.

Five elastic constants:

\[
\begin{align*}
C^{1111} &= C^{2222} (\equiv c^{1111} = c^{2222}), \\
C^{1122} &= C^{3333} (\equiv c^{3333}), \\
2C^{1212} &= 2c^{1212} = (c^{1111} - c^{1122}).
\end{align*}
\]

Three piezoelectric constants:

\[
\begin{align*}
C^{4311} &= C^{4322} (\equiv e^{311} = e^{322}), \\
C^{4333} &= C^{4413} (\equiv e^{333}), \\
C^{5113} &= C^{5242} (\equiv e^{113} = e^{223}).
\end{align*}
\]

Three piezomagnetic constants:

\[
\begin{align*}
C^{5311} &= C^{5322} (\equiv q^{311} = q^{322}), \\
C^{5333} &= C^{5513} (\equiv q^{333}), \\
C^{5353} &= C^{5533} (\equiv q^{333}).
\end{align*}
\]

Two dielectric permittivity constants:

\[
\begin{align*}
C^{4141} &= C^{4242} (\equiv -\kappa^{11} = -\kappa^{22}), \\
C^{4343} &= C^{4434} (\equiv -\kappa^{33}).
\end{align*}
\]

Two ME constants:

\[
\begin{align*}
C^{5141} &= C^{5242} (\equiv -\alpha^{11} = -\alpha^{22}), \\
C^{5343} &= C^{5434} (\equiv -\alpha^{33}).
\end{align*}
\]

Two magnetic permittivity constants:

\[
\begin{align*}
C^{5151} &= C^{5252} (\equiv -\mu^{11} = -\mu^{22}), \\
C^{5353} &= C^{5553} (\equiv -\mu^{33}).
\end{align*}
\]

5.1 Connectivity in parallel

Using formula (5.5), the effective coefficients for this composite material are the following.

Nine elastic effective constants:

\[
\begin{align*}
\hat{c}^{1111} &= \langle c^{1111} \rangle - \langle c^{1122} \rangle^2 / c^{2222} + \langle c^{1122} \rangle^2 / (1 / c^{2222}), \\
\hat{c}^{1122} &= \langle c^{1122} \rangle c^{2222} / (1 / c^{2222}), \\
\hat{c}^{1133} &= \langle c^{1133} \rangle - \langle c^{1122} \rangle c^{2233} / c^{2222} + \langle c^{1122} \rangle c^{2222} / (1 / c^{2222}), \\
\hat{c}^{2222} &= 1 / (1 / c^{2222}), \\
\hat{c}^{3333} &= \langle c^{3333} \rangle - \langle c^{3322} \rangle^2 / c^{2222} + \langle c^{3322} \rangle^2 / (1 / c^{2222}), \\
\hat{c}^{1313} &= \langle c^{1313} \rangle, \\
\hat{c}^{1212} &= 1 / (1 / c^{1212}), \\
\hat{c}^{2323} &= e_1^t (M_{23}^{-1})^{-1} e_1. \tag{5.7}
\end{align*}
\]
Five piezoelectric effective constants:
\[
\tilde{e}^{113} = \langle e^{113} \rangle, \quad \tilde{e}^{223} = e_2' (M_{23}^{-1})^{-1} e_1,
\]
\[
\tilde{e}^{311} = \langle e^{311} \rangle + \langle e^{311}/c_{2222} \rangle (c_{2211}/c_{2222})/(1/c_{2222}) - \langle e^{311} c_{2211}/c_{2222} \rangle,
\]
\[
\tilde{e}^{322} = \langle e^{322}/c_{2222} \rangle/(1/c_{2222}),
\]
\[
\tilde{e}^{333} = \langle e^{333} \rangle + \langle e^{322}/c_{2222} \rangle (c_{2233}/c_{2222})/(1/c_{2222}) - \langle e^{322} c_{2233}/c_{2222} \rangle. \tag{5.8}
\]

Three dielectric permittivity effective constants:
\[
\tilde{\kappa}^{11} = \langle \kappa^{11} \rangle, \quad \tilde{\kappa}^{22} = -e_2' (M_{23}^{-1})^{-1} e_2,
\]
\[
\tilde{\kappa}^{33} = \langle \kappa^{33} \rangle + \langle e^{322}/c_{2222} \rangle^2/(c_{2222}) - \langle e^{322}/c_{2222} \rangle^2/(1/c_{2222}). \tag{5.9}
\]

Five piezomagnetic effective constants:
\[
\tilde{q}^{113} = \langle q^{113} \rangle, \quad \tilde{q}^{223} = e_3' (M_{23}^{-1})^{-1} e_1,
\]
\[
\tilde{q}^{311} = \langle q^{311} \rangle + \langle q^{311}/c_{2222} \rangle (c_{2211}/c_{2222})/(1/c_{2222}) - \langle q^{311} c_{2211}/c_{2222} \rangle,
\]
\[
\tilde{q}^{322} = \langle q^{322}/c_{2222} \rangle/(1/c_{2222}),
\]
\[
\tilde{q}^{333} = \langle q^{333} \rangle + \langle q^{322}/c_{2222} \rangle (c_{2233}/c_{2222})/(1/c_{2222}) - \langle q^{322} c_{2233}/c_{2222} \rangle. \tag{5.10}
\]

Three ME effective constants:
\[
\tilde{\alpha}^{11} = \langle \alpha^{11} \rangle, \quad \tilde{\alpha}^{22} = -e_2' (M_{23}^{-1})^{-1} e_2,
\]
\[
\tilde{\alpha}^{33} = \langle \alpha^{33} \rangle + \langle q^{322}/e_{2222} \rangle - \langle q^{322}/c_{2222} \rangle (e^{322}/c_{2222})/(1/c_{2222}). \tag{5.11}
\]

Three magnetic permittivity constants:
\[
\tilde{\mu}^{11} = \langle \mu^{11} \rangle, \quad \tilde{\mu}^{22} = -e_3' (M_{23}^{-1})^{-1} e_3,
\]
\[
\tilde{\mu}^{33} = \langle \mu^{33} \rangle + \langle q^{322}/c_{2222} \rangle^2/(1/c_{2222}) - \langle q^{322}/c_{2222} \rangle^2/(1/c_{2222}). \tag{5.12}
\]

Here, \( e_i \) (i = 1, 2, 3) are the vectors of the standard orthonormal basis for the Euclidean space \( \mathbb{R}^3 \) and \( M_{23}^{-1} \) is the inverse matrix of
\[
M_{23} = \begin{pmatrix}
  c_{2323} & e_{223} & q_{223} \\
  e_{223} & -\kappa^{22} & -\alpha^{22} \\
  q_{223} & -\alpha^{22} & -\mu^{22}
\end{pmatrix}.
\]

For this class of laminated composites, connected in parallel, the corresponding homogenized material behaves as a magneto-electro-elastic material with orthorhombic symmetry (2 mm). The effective elastic, piezoelectric and dielectric constants (expressions (5.7)–(5.9)) are exactly the same as formulae (41)–(43) in (23, pp. 537–538).

From (5.7)_9, (5.9)_2, (5.10)_2, (5.11)_2 and (5.12)_2, the expression below can be found:
\[
\tilde{M}_{23} = (M_{23}^{-1})^{-1}, \quad \text{where} \quad \tilde{M}_{23} = \begin{pmatrix}
  \tilde{c}_{2323} & \tilde{e}_{223} & \tilde{q}_{223} \\
  \tilde{e}_{223} & -\tilde{\kappa}^{22} & -\tilde{\alpha}^{22} \\
  \tilde{q}_{223} & -\tilde{\alpha}^{22} & -\tilde{\mu}^{22}
\end{pmatrix}.
\]
Consequently, the following relations can be derived:

$$\frac{\bar{c}^{2323}}{\Delta_{11}} = -\frac{\bar{e}^{332}}{\Delta_{12}} = -\frac{\bar{k}^{22}}{\Delta_{22}} = \frac{q^{232}}{\Delta_{13}} = \frac{\bar{a}^{22}}{\Delta_{23}} = -\frac{\bar{\mu}^{22}}{\Delta_{33}} = \frac{1}{\Delta},$$

where $\Delta$ is the determinant of the matrix $\langle M_{23}^{-1} \rangle$ and $\Delta_{ij}$ is the minor obtained by excluding the $i$th row and $j$th column. From (5.13), one can observe that if one of the six effective coefficients is known, then it is possible to calculate the other ones. It is interesting to note that for the particular case when $q^{233} = 0$ and $a^{22} = 0$, the following relationship between the effective coefficients $\bar{c}^{2323}$, $\bar{e}^{332}$ and $\bar{k}^{22}$ can be obtained:

$$\frac{\bar{c}^{2323}}{(c^{2323}/\tau)} = \frac{\bar{e}^{332}}{(e^{332}/\tau)} = \frac{\bar{k}^{22}}{(k^{22}/\tau)} = \frac{1}{\Delta},$$

with $\tau = c^{2323}k^{22} + (e^{2323})^2$ and $\Delta = (c^{2323}/\tau)(k^{22}/\tau) + (e^{2323}/\tau)^2$.

The effective properties involved in relation (5.14) are the same as those that appear in (2.13d,g,h) of (29, p. 37). See also their similarity with formulae (30) in (30, p. 83) derived there from the 1D example of piezoelectric constituents.

### 5.2 Connectivity in series

In this example, we assume that the laminated composite possesses the same periodic properties as in the previous example, but the distribution of the cells is periodic along the $y_3$-axis. The axes of symmetry of each layer are parallel to each other and the $y_3$-axis is perpendicular to the layering. Then, using formula (5.5) (interchanging in these expressions the indices 2 by 3) the following effective coefficients can be derived.

Five elastic effective constants:

$$\bar{c}^{1111} = \langle c^{1111} \rangle + \langle C_1'M_{33}^{-1}\rangle\langle M_{33}^{-1}\rangle^{-1}\langle M_{33}^{-1}C_1 \rangle - \langle C_1'M_{33}^{-1}C_1 \rangle = \bar{c}^{2222},$$

$$\bar{c}^{1122} = \langle c^{1122} \rangle + \langle C_1'M_{33}^{-1}\rangle\langle M_{33}^{-1}\rangle^{-1}\langle M_{33}^{-1}C_1 \rangle - \langle C_1'M_{33}^{-1}C_1 \rangle,$$

$$\bar{c}^{1133} = \langle C_1'M_{33}^{-1}\rangle^{-1}e_1 = e^{2323},$$

$$\bar{c}^{1333} = \langle C_1'M_{33}^{-1}\rangle^{-1}e_1 = e^{3333} = e_1'(M_{33}^{-1})^{-1}e_1.$$

$$\bar{c}^{1313} = 1/\langle 1/c^{1313} \rangle = c^{2323},$$

$$\bar{c}^{1212} = (\bar{c}^{1111} - \bar{c}^{2222})/2.$$  

Three piezoelectric effective constants:

$$\bar{e}^{113} = \langle e^{113}/c^{1313} \rangle/\langle 1/c^{1313} \rangle = \bar{e}^{223},$$

$$\bar{e}^{311} = e_1'(M_{33}^{-1})^{-1}(M_{33}^{-1}C_1) = \bar{e}^{322},$$

$$\bar{e}^{333} = e_2'(M_{33}^{-1})^{-1}e_2.$$  

Two dielectric effective constants:

$$\bar{\varepsilon}^{11} = \langle \varepsilon^{11} \rangle + ((e^{113})^2/c^{1313}) - \langle e^{113}/c^{1313} \rangle^2/\langle 1/c^{1313} \rangle = \bar{\varepsilon}^{22},$$

$$\bar{\varepsilon}^{33} = -e_2'(M_{33}^{-1})^{-1}e_2.$$  

(5.17)
Three piezomagnetic effective constants:
\[
\begin{align*}
\tilde{q}^{113} &= \langle q^{113}/c^{1313} \rangle / (1/c^{1313}) = q^{223}, \\
\tilde{q}^{311} &= \tilde{e}_3^t (M_{33}^{-1})^{-1} \langle M_{33}^{-1} \rangle = \tilde{q}^{322}, \\
\tilde{q}^{333} &= \tilde{e}_3^t (M_{33}^{-1})^{-1} e_1. 
\end{align*}
\]

(5.18)

Two ME effective constants:
\[
\begin{align*}
\tilde{\alpha}^{11} &= \langle \alpha^{11} \rangle + \langle e^{113} q^{113}/c^{1313} \rangle - \langle e^{113}/c^{1313} \rangle \langle q^{113}/c^{1313} \rangle / (1/c^{1313}) = \tilde{\alpha}^{22}, \\
\tilde{\alpha}^{33} &= - \tilde{e}_3^t (M_{33}^{-1})^{-1} e_2.
\end{align*}
\]

(5.19)

Two magnetic permittivity constants:
\[
\begin{align*}
\tilde{\mu}^{11} &= \langle \mu^{11} \rangle + \langle (q^{113})^2/c^{1313} \rangle - \langle q^{113}/c^{1313} \rangle^2 / (1/c^{1313}) = \tilde{\mu}^{22}, \\
\tilde{\mu}^{33} &= - \tilde{e}_3^t (M_{33}^{-1})^{-1} e_3.
\end{align*}
\]

(5.20)

Here, \(C_1^t = (c^{113} e^{311} q^{311})\) and \(M_{33}^{-1}\) is the inverse matrix of
\[
M_{33} = \begin{pmatrix}
c^{3333} & e^{333} & q^{333} \\
e^{333} & -\kappa^{33} & -\alpha^{33} \\
q^{333} & -\alpha^{33} & -\mu^{33}
\end{pmatrix}.
\]

In this case, the global transversely isotropic symmetry is preserved as expected. It is possible to observe that, from formulae (5.15), (5.16)3, (5.17)2, (5.18)3, (5.19)2 and (5.20)2,
\[
\begin{align*}
\bar{c}^{3333} &= - \bar{c}^{333}, \\
\bar{\kappa}^{33} &= - \bar{\kappa}^{33}, \\
\bar{q}^{333} &= \bar{q}^{333}, \\
\bar{\alpha}^{33} &= - \bar{\alpha}^{33}, \\
\bar{\mu}^{33} &= - \bar{\mu}^{33},
\end{align*}
\]

\[(5.21)\]

where \(\Delta^t\) is the determinant of the matrix \(M_{33}^{-1}\) and \(\Delta_{ij}^t\) is the minor obtained by excluding the \(i\)th row and \(j\)th column. From (5.21), it is noted that if one of the six effective coefficients is known, then it is possible to calculate the other ones.

From (5.15)–(5.20), it is possible to obtain a set of interesting relations (see, for instance, (31, p. 162)) between the effective coefficients and the averages of the corresponding constitutive properties. Some of these relations are
\[
\begin{align*}
\bar{M}_{33} &= (\bar{M}_{33}^{-1})^{-1}, \\
\bar{C}_1^t M_{33}^{-1} &= (C_1^t M_{33}^{-1}), \\
\bar{c}^{111i} &= \tilde{C}_1^t \tilde{M}_{33}^{-1} \tilde{C}_1 = (c^{111i} - C_1^t M_{33}^{-1} C_1), \\
\bar{\kappa}^{11} &= \langle e^{113} \rangle^2 / \bar{c}^{1313} = (\kappa^{11} + \langle e^{113} \rangle^2 / c^{1313}), \\
\bar{\mu}^{11} &= \langle q^{113} \rangle^2 / \bar{c}^{1313} = (\mu^{11} + \langle q^{113} \rangle^2 / c^{1313}), \\
\bar{\alpha}^{11} &= \bar{c}^{113} q^{113} / \bar{c}^{1313} = (\alpha^{11} + e^{113} q^{113} / c^{1313}),
\end{align*}
\]
where \( \tilde{\mathbf{C}}_1 = (\tilde{c}^{1133} \tilde{e}^{311} \tilde{q}^{311}) \) and

\[
\tilde{\mathbf{M}}_{33} = \begin{pmatrix}
\tilde{c}^{3333} & \tilde{e}^{3333} & \tilde{q}^{3333} \\
\tilde{e}^{3333} & -\tilde{\kappa}^{33} & -\tilde{\alpha}^{33} \\
\tilde{q}^{3333} & -\tilde{\alpha}^{33} & -\tilde{\mu}^{33}
\end{pmatrix}.
\]

6. Numerical results and comments

All the computations reported in this section were implemented in MATLAB code, using our own software, by a direct programming of the formulae for multilaminated media presented in the above section. Material properties used in the calculations are listed in Table 1 and were taken from (12) and (32).

First, the code was applied to calculate all the effective magneto-electro-elastic coefficients of a two-laminated composite, in series connection, consisting of piezoelectric BaTiO\(_3\) and piezomagnetic CoFe\(_2\)O\(_4\) as a function of BaTiO\(_3\) volume fraction. In this particular case, all the corresponding

![Fig. 2](image-url)

**Fig. 2** Effective elastic constants of a parallel-connected CoFe\(_2\)O\(_4\)–BaTiO\(_3\) laminated composite as a function of BaTiO\(_3\) volume fraction. In Fig. 2(b), note the strong nonlinear behaviour of \( c^{2323} \)
Table 1  Magneto-electro-elastic material properties

<table>
<thead>
<tr>
<th></th>
<th>$c_{1111}$</th>
<th>$c_{1122}$</th>
<th>$c_{1133}$</th>
<th>$c_{3333}$</th>
<th>$c_{2323}$</th>
<th>$c_{1212}$</th>
<th>$\kappa_{11}$</th>
<th>$\kappa_{33}$</th>
</tr>
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<td><strong>Units</strong></td>
<td>GPa</td>
<td>GPa</td>
<td>GPa</td>
<td>GPa</td>
<td>GPa</td>
<td>C²/Nm²</td>
<td>C²/Nm²</td>
<td></td>
</tr>
<tr>
<td><strong>BaTiO₃</strong></td>
<td>166</td>
<td>77</td>
<td>78</td>
<td>162</td>
<td>43</td>
<td>44·5</td>
<td>1·2 × 10⁻⁹</td>
<td>12·6 × 10⁻⁹</td>
</tr>
<tr>
<td><strong>CoFe₂O₄</strong></td>
<td>286</td>
<td>173</td>
<td>170</td>
<td>269·5</td>
<td>45·3</td>
<td>56·5</td>
<td>0·08 × 10⁻⁹</td>
<td>0·093 × 10⁻⁹</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$e_{223}$</th>
<th>$e_{311}$</th>
<th>$e_{333}$</th>
<th>$q_{223}$</th>
<th>$q_{311}$</th>
<th>$q_{333}$</th>
<th>$\mu_{11}$</th>
<th>$\mu_{33}$</th>
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<td><strong>Units</strong></td>
<td>C/m²</td>
<td>C/m²</td>
<td>C/m²</td>
<td>m/A</td>
<td>m/A</td>
<td>m/A</td>
<td>Ns²/C²</td>
<td>Ns²/C²</td>
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<tr>
<td><strong>BaTiO₃</strong></td>
<td>11·6</td>
<td>18·6</td>
<td>11·6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5 × 10⁻⁶</td>
<td>10 × 10⁻⁶</td>
</tr>
<tr>
<td><strong>CoFe₂O₄</strong></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>550</td>
<td>580·3</td>
<td>699·7</td>
<td>157 × 10⁻⁶</td>
<td>157 × 10⁻⁶</td>
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</tbody>
</table>

Fig. 3  Effective (a) piezoelectric and (b) piezomagnetic constants of a parallel-connected CoFe₂O₄–BaTiO₃ laminated composite as a function of BaTiO₃ volume fraction. Note the significant nonlinear behaviour of $e_{223}$ and $q_{223}$ constants.
Fig. 4 Effective (a) dielectric and (b) magnetic constants of a parallel-connected CoFe$_2$O$_4$–BaTiO$_3$ laminated composite as a function of BaTiO$_3$ volume fraction. Note the nonlinearity exhibited by $\kappa^{22}$ and $\mu^{22}$ constants.
In Figs 2 to 5, all effective properties involved in relation (5.13) show highly nonlinear behaviour with respect to the BaTiO$_3$ volume fraction (see, for instance, solid lines in Figs 2(d), 3(a,b), 4(a,b) and 5(a) which correspond to $c^{2323}$, $\tilde{e}^{223}$, $\tilde{q}^{223}$, $\tilde{\kappa}^{22}$, $\tilde{\mu}^{22}$ and $\tilde{\alpha}^{22}$, respectively). Also, the ME effective coefficient $\tilde{\alpha}^{33}$ changes nonlinearly with volume fraction. This is an interesting coefficient as it reveals the ME effect in the composite even if none of the individual phases exhibit it. It is a consequence of the interaction between the piezoelectric and piezomagnetic phases which can be seen in formula (5.11)$_3$. All other effective coefficients are essentially linear with volume fraction. It is interesting to mention that almost all overall properties have the same trends which appear reported, for fibrous composite, in Figs 1 to 5 of (12), with the exception of the effective ME coefficient $\tilde{\alpha}^{22}$ which has the same trend as the $\tilde{\alpha}^{33}$ of laminated composite connected in series (see (12, Fig. 6b)). These analogies could be a consequence of the fact that the two-laminated media connected in parallel behaves as a limit case of a fibrous composite in the $y_3$-direction, whereas it behaves like a binary laminated composite connected in series in the $y_2$-direction.

**Fig. 5** Effective ME constants of a parallel-connected CoFe$_2$O$_4$–BaTiO$_3$ laminated composite as a function of BaTiO$_3$ volume. Note the ME effect shown by $\tilde{\alpha}^{22}$ and $\tilde{\alpha}^{33}$ constants
7. Conclusions

In this paper, we have rigorously established the limiting equations modelling the behaviour of magneto-electro-elastic periodic structures, that is, we have explicitly derived the local problems, (4.4)–(4.7), and the general formulae (4.10) for the homogenized coefficients of the elastic, dielectric, piezoelectric, magnetic and piezomagnetic tensors. Some important mathematical results of homogenization are rigorously established, for instance, the conservation of the classical properties of symmetry and positivity for the effective coefficients (Lemma 4.4), existence and uniqueness theorem (Theorem 4.5) and equivalences of convergences when the geometrical parameter goes to zero (Lemma 4.6 and Corollary 4.7). The same framework is useful in the thermo-piezoelectric and thermo-magneto-electro-elastic models (see, for instance, (27)). It is necessary to remark that the homogenization process described in this work is fine for regions far enough away from the boundary so that its effect is not felt because near boundaries the material will not behave as an effective material with homogenized coefficients. To do homogenization properly on bounded domains, the so-called boundary-layer technique must be used (24, 33, 34). Although the method used here applies to statics, its generalization to dynamical problems is quite straightforward in the quasi-static regime of a very long wavelength as compared with the size of the periodic cell. Recently, Parnell and Abrahams (35) have studied the problem of anti-plane shear waves in periodic fibre-reinforced composites formally employing the asymptotic homogenization method.

Based on our homogenization model, closed-form expressions for effective coefficients of multilaminated magneto-electro-elastic composites, with perfect bonding conditions at the interfaces, were derived ((5.5) and (5.6)). Such general formulae are specified for the important case of transversely isotropic constituents and all effective properties are explicitly given for two examples of periodic laminated composites, with any finite number of layers in the unit periodic cell (see (5.7)–(5.11) and (5.15)–(5.20)). In these examples, two interesting identities, that is, $\tilde{M}_{23} \equiv (M_{23}^{-1})^{-1}$ and $\tilde{M}_{33} \equiv (M_{33}^{-1})^{-1}$, were obtained. Based on these identities, two chains of equalities ((5.13) and (5.21)) relating the ME effective moduli ($\tilde{\alpha}_{22}$ and $\tilde{\alpha}_{33}$) to five other effective coefficients, each, were obtained. The nonlinear behaviour, which is observed from the numerical experiments given in section 6, is exhibited by the effective coefficients involved in (5.13) and (5.21).

Finally, formula (5.5) is a general expression of effective coefficients for multilaminated magneto-electro-elastic composites with a unit periodic cell having any finite number of layers. As an example, analytical results are given for some important particular cases, for instance, a multilaminated transversely isotropic magneto-electro-elastic composite which in the particular case of a two-laminated composite is numerically the same as that reported by Li and Dunn (12) who used an approximate method.

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